

# From Lattice Gauge Theories to Hydrogen Atoms

Manu Mathur<sup>a</sup>, T. P. Sreeraj<sup>a</sup>

<sup>a</sup>*S. N. Bose National Centre for Basic Sciences, Salt Lake, JD Block, Sector 3, Kolkata 700098, India*

## Abstract

We construct canonical transformations to obtain a complete and most economical realization of the physical Hilbert space  $\mathcal{H}^p$  of pure  $SU(2)_{2+1}$  lattice gauge theory in terms of Wigner coupled Hilbert spaces of hydrogen atoms. One hydrogen atom is assigned to every plaquette of the lattice. A complete orthonormal description of the Wilson loop basis in  $\mathcal{H}^p$  is obtained by all possible angular momentum Wigner couplings of hydrogen atom energy eigenstates  $|n\ l\ m\rangle$  describing electric fluxes on the loops. The  $SU(2)$  gauge invariance implies that the total angular momenta of all hydrogen atoms vanish. The canonical transformations also enable us to rewrite the Kogut-Susskind Hamiltonian in terms of fundamental Wilson loop operators and their conjugate electric fields. The resulting loop Hamiltonian has a global  $SU(2)$  invariance and a simple weak coupling ( $g^2 \rightarrow 0$ ) continuum limit. The canonical transformations leading to the loop Hamiltonian are valid for any  $SU(N)$ . The ideas and techniques can also be extended to higher dimension.

## 1. Introduction

The idea that gauge theories should be reformulated as theory of loops and strings without any color degrees of freedom is old [1] and refuses to die. One hopes that eventually some appropriate loop description of gauge theories will provide a natural and elegant framework to compute low energy QCD effects leading to a better understanding of non-perturbative issues like color confinement. Since the work of Ashtekar, loops carrying  $SU(2)$  fluxes have also found their relevance in quantum gravity where they describe quantum excitations of geometry [4]. In condensed matter physics [5], many effective models are in terms of non-abelian gauge theories. In view of above, the importance of developing new ideas and techniques to understand gauge theories better requires no emphasis. In recent past, quest to realize simplest non-abelian  $SU(2)$  lattice gauge theory Hamiltonian dynamics using cold atomic gases in optical lattices [6] as well as  $SU(2)$  tensor network ideas [7] to explore the interesting part of the physical Hilbert space  $\mathcal{H}^p$  are important and exciting new developments in this direction.

In this letter, we show that the gauge invariant physical or loop Hilbert space  $\mathcal{H}^p$  of pure  $SU(2)$  lattice gauge theory can be completely and most economically realized in terms of the Wigner coupled bound energy eigenstates  $|n\ l\ m\rangle$  of hydrogen atoms associated with the plaquettes of the lattice as shown in figure (1). This Wigner coupled hydrogen atom basis describes quantized  $SU(2)$  loop electric fluxes in terms of  $(n, l, m)$  and is orthonormal as well as complete in  $\mathcal{H}^p$ . Therefore, our description of loop dynamics is com-

pletely free of the notorious Mandelstam constraints<sup>1</sup>. We rewrite the Kogut-Susskind Hamiltonian in terms of the fundamental plaquette loop operators and conjugate loop electric fields. We show that this loop Hamiltonian has a global  $SU(2)$  symmetry and there are no gauge fields. Further, the loop Hamiltonian has a simple weak coupling ( $g^2 \rightarrow 0$ ) limit. We work in two space dimension on a finite lattice with  $\mathcal{N} = (N+1) \times (N+1)$  sites,  $\mathcal{L}$  links and  $\mathcal{P}$  plaquettes satisfying:  $\mathcal{L} = \mathcal{P} + (\mathcal{N} - 1)$ . We choose open boundary conditions. A lattice site is denoted by  $(x, y)$  with  $x, y = 0, 1, \dots, N$ . There are no static or dynamical charges. All “hydrogen atom solutions” of the local Gauss law constraints are obtained in the charge zero sector.

Following Fock [2], we describe  $\mathcal{P}$  hydrogen atoms in figure (1) on their momentum hypersphere  $S^3$  so that their hidden  $SU(2) \times SU(2)$  symmetries become manifest (see section (2), equation (4)). On the other hand, we construct  $SU(2)$  ( $SU(N)$ ) canonical transformations which fuse  $\mathcal{L}$  Kogut-Susskind link operators into  $\mathcal{P}$  fundamental plaquette loop operators, each enclosing one of the  $\mathcal{P}$  plaquettes as shown in figure (5-a). The equivalence of the gauge theory and hydrogen atom Hilbert spaces has its origin in the identification of  $SU(2)$  group manifold  $S^3$  associated with each plaquette loop holonomy with the  $S^3$  of the corresponding hydrogen atom.

The plan of the paper is as follows. In section (2) we briefly review hydrogen atom Hilbert space in a language [3] which helps us later in establishing its connection with

<sup>1</sup>In fact, these constraints have been major obstacles in the loop formulation because of their non-local nature [8, 10]. They have been extensively discussed in the past [10] in the context of  $SU(2)$  gauge theory leading to spin networks as their solutions [8, 9, 10]. However, loop dynamics is extremely complicated in the spin network basis [8, 9].

Email addresses: manu@bose.res.in (Manu Mathur), sreerajtp@bose.res.in (T. P. Sreeraj)

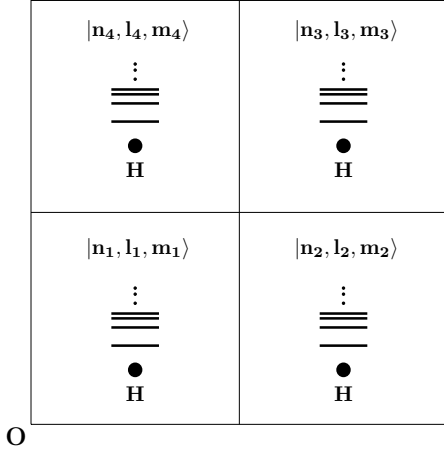


Figure 1: One Hydrogen atom, denoted by  $\bullet$ , is assigned to each plaquette. The energy eigenstates  $|n_p l_p m_p\rangle$  Wigner couple together (19) with vanishing total angular momenta to form a basis in the physical Hilbert space  $\mathcal{H}^p$  of pure  $SU(2)$  lattice gauge theory.

lattice gauge theory. The section (3) discusses the canonical transformations which fuse all Kogut-Susskind  $SU(N)$  link operators into mutually independent and fundamental Wilson loop operators. Having discussed hydrogen atoms and loop formulation separately, in section (4) we establish the exact equivalence of the Hilbert spaces of hydrogen atoms and  $SU(2)$  lattice gauge theory. In section (5) we construct loop Hamiltonian and briefly discuss loop dynamics. We end the paper with a short discussion on our results and some of the possible future directions.

## 2. Hydrogen Atom

As shown by Pauli [2] in 1926, the hydrogen atom can be elegantly solved using group theory which exploits manifest rotational and hidden Runge Lenz symmetries generated by angular momentum ( $\vec{L}$ ) and Laplace Runge Lenz ( $\vec{A}$ ) operators<sup>2</sup> respectively. These generators commute with the hydrogen atom Hamiltonian and satisfy  $\vec{L} \cdot \vec{A} = 0$ . This leads to  $SU(2) \otimes SU(2)$  symmetry algebra generated by  $\vec{J}_\pm \equiv 1/2(\vec{L} \pm \vec{A})$  on the bound states of hydrogen atom ( $E < 0$ ) [3]. Further, as  $[J_+^a, J_-^b] = 0$  and  $\vec{J}_+^2 = \vec{J}_-^2 \equiv \vec{J}^2$ , the two equivalent complete set of commuting operators (CSCO) are  $[\vec{J}^2, J_+^z, J_-^z]$  (CSCO-I) in the decoupled basis and  $[\vec{J}^2, \vec{L}^2, \vec{L}_z]$  (CSCO-II) in the coupled basis. Following Wybourne [3], we define<sup>3</sup>:

$$J_-^a \equiv a^\dagger \left( \frac{\sigma^a}{2} \right) a; \quad J_+^a \equiv b^\dagger \left( \frac{\sigma^a}{2} \right) b. \quad (1)$$

<sup>2</sup>We follow Wybourne [3] for hydrogen atom discussions. The Runge Lenz vector has been scaled by  $\frac{1}{\sqrt{-2H}}$ .

<sup>3</sup> $J_\pm^a$  should not be confused with the raising and lowering angular momentum operators.

In (1)  $(a_1^\dagger, a_2^\dagger)$  and  $(b_1^\dagger, b_2^\dagger)$  represent  $SU(2)$  doublets of Schwinger boson creation operators,  $\sigma^a$  ( $a=1,2,3$ ) are the Pauli matrices. The condition  $\vec{J}_-^2 = \vec{J}_+^2$  implies  $N_a = N_b$  where  $N_a = a^\dagger \cdot a$  and  $N_b = b^\dagger \cdot b$  are the total number operators. The orthonormal and complete basis diagonalizing CSCO-I is given by [3]:

$$|j = j_- = j_+, m_-, m_+\rangle = |j, m_-\rangle \otimes |j, m_+\rangle \quad (2)$$

$$|j_- = j, m_-\rangle \equiv \frac{(a_1^\dagger)^{(j+m_-)} (a_2^\dagger)^{(j-m_-)}}{(j+m_-)!(j-m_-)!} |0\rangle$$

$$|j_+ = j, m_+\rangle \equiv \frac{(b_1^\dagger)^{(j+m_+)} (b_2^\dagger)^{(j-m_+)}}{(j+m_+)!(j-m_+)!} |0\rangle.$$

The other equivalent coupled hydrogen atom basis diagonalizing the CSCO-II is given by:

$$|n l m\rangle \equiv \sum_{m_-, m_+} C_{j m_-, j m_+}^{l, m} |j, m_-, m_+\rangle \quad (3)$$

In (3),  $n \equiv (2j+1) = 1, 2, \dots; l = 0, 1, 2, \dots, (n-1); m = -l, \dots, +l; C_{j m', j m}^{l, m}$  are the Clebsch-Gordan coefficients. The hydrogen atom states  $|n l m\rangle$  are eigenstates of  $J^2, L^2, L_z$  and also of the Hamiltonian with energy [3]  $E \sim -1/n^2$ . For later purpose, it is convenient to graphically represent the hydrogen atom states  $|n l m\rangle$  in (3) by a tadpole in figure (4).

As shown by Fock, the above  $SU(2) \otimes SU(2)$  symmetry for bound states ( $p_0^2 \equiv -2E > 0$ ) becomes manifest if we transcribe the hydrogen atom dynamics on a hypersphere  $S^3 : (q_0, \vec{q}; q_0^2 + \vec{q}^2 = 1)$  embedded in  $R^4 : (p_0, p_1, p_2, p_3)$  through a stereographic projection:

$$q_0 \equiv \frac{(p_0^2 - \vec{p}^2)}{(p_0^2 + \vec{p}^2)}, \quad \vec{q} \equiv \frac{2p_0 \vec{p}}{(p_0^2 + \vec{p}^2)}, \quad (4)$$

$$\Omega_H(q_0, \vec{q}) \equiv q_0 \sigma_0 + i \vec{q} \cdot \vec{\sigma}, \quad q_0^2 + \vec{q}^2 = 1.$$

Above  $\sigma_0, \vec{\sigma}$  are the identity, Pauli matrices respectively. The mapping (4) enables us to transform [2] momentum space hydrogen atom Schrodinger equation into the integral equation of the 4-dimensional spherical harmonics  $Y_{n,l,m}(\Omega_H)$  representing a free particle on  $S^3$ . It was later shown by Bargmann [2] that  $(L_1, L_2, L_3)$  and  $(A_1, A_2, A_3)$  correspond to rotations in  $(q_2 q_3), (q_1 q_3), (q_1 q_2)$  and  $(q_0 q_1), (q_0 q_2), (q_0 q_3)$  planes respectively making  $SU(2) \otimes SU(2)$  symmetry of hydrogen atom manifest.

We now construct iterative canonical transformations which make all gauge degrees of freedom decouple from  $\mathcal{H}^p$  by systematically fusing the Kogut-Susskind [11] link flux operators into a complete and mutually independent set of plaquette loop operators. These loop operators (11),(14) in turn are then associated with hydrogen atoms. To keep the presentation simple and short, we always illustrate the ideas and techniques on a single plaquette and then generalize the results to the entire lattice. We have chosen space dimension  $d = 2$  for the same reasons.

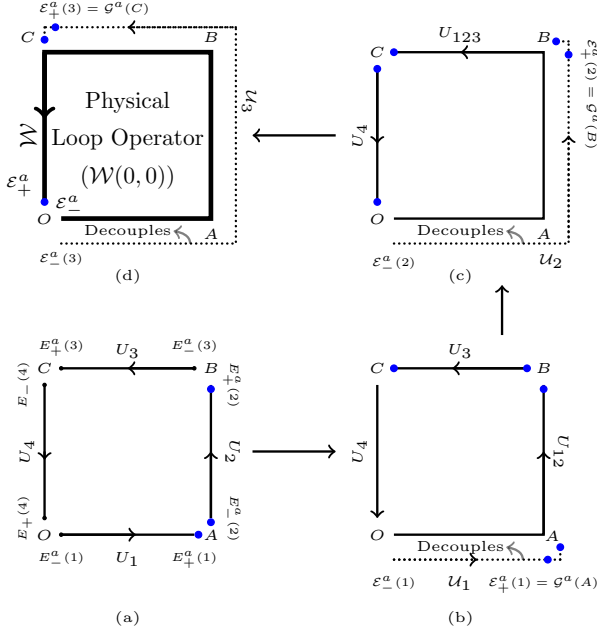


Figure 2: Three canonical transformations on the four link flux operators of a plaquette leading to a single plaquette loop flux operator  $\mathcal{W}_{\alpha\beta}$ . The electric fields involved in the canonical transformations are represented by  $\bullet$ . Two  $\bullet$  at the end of flux operators  $\mathcal{U}(1), \mathcal{U}(2)$  and  $\mathcal{U}(3)$  represent their electric fields which are also the Gauss law operators  $\mathcal{G}^a$  at A, B and C respectively. Only the loop operator ( $\mathcal{W}_{\alpha\beta}, \mathcal{E}_{\mp}$ ) are relevant, the remaining are unphysical and to be ignored.

### 3. Canonical Transformations: From Links to Loops

#### 3.1. Single Plaquette Case

We start with a plaquette OABC with 4 Kogut-Susskind SU(2) flux operators [11]  $U_I, I = 1, 2, 3, 4$  attached to 4 sides. The left, right rotations on  $U_I$  are generated by the left, right electric fields  $E_-^a(l), E_+^a(l)$  ( $a=1,2,3$ ) respectively as shown in figure (2-a). The basic quantization rules are:

$$\begin{aligned} [E_+^a, U_{\alpha\beta}] &= \left( U \frac{\sigma^a}{2} \right)_{\alpha\beta} \Rightarrow [E_+^a, E_+^b] = i\epsilon^{abc} E_+^c \\ [E_-^a, U_{\alpha\beta}] &= - \left( \frac{\sigma^a}{2} U \right)_{\alpha\beta} \Rightarrow [E_-^a, E_-^b] = i\epsilon^{abc} E_-^c \end{aligned} \quad (5)$$

As operators on different links commute, we have suppressed the link index  $l$  in (5). One can check that  $E_+^a = -R_{ab}(U^\dagger)E_-^b$  where  $R_{ab}(U) = \frac{1}{2}\text{Tr}(\sigma^a U \sigma^b U^\dagger)$  is SO(3) rotation matrix implying:  $(\vec{E}_-)^2 = (\vec{E}_+)^2$ ,  $[E_-^a, E_+^b] = 0$ , on every links.

We now make canonical transformations to fuse  $\{U_1, U_2, U_3, U_4\}$  into 3 unphysical string flux operators  $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$  and a physical Wilson plaquette loop operator<sup>4</sup>  $\mathcal{W}$  around OABC as shown in figure 2. The corre-

sponding left, right string and loop electric fields are denoted by  $\mathcal{E}_{\mp}^a(I)$  ( $I = 1, 2, 3$ ) and  $\mathcal{E}_{\mp}^a$  respectively as shown in figure (2-a,b,c,d). The first canonical transformation fusing  $U_1$  and  $U_2$ , is:

$$\begin{aligned} U_1 &\rightarrow \mathcal{U}_1 \equiv U_1, \quad U_2 \rightarrow U_{12} \equiv U_1 U_2, \\ \Rightarrow \mathcal{E}_+^a(1) &= E_+^a(1) + E_-^a(2), \quad E_+^a(12) = E_+^a(2). \end{aligned} \quad (6)$$

Note that the new pairs  $\{\mathcal{U}(1), \mathcal{E}_+(1)\}$  and  $\{U_{12}, E_+(12)\}$  are canonical satisfying (5) and also mutually independent. The SU(2) Gauss law at the corner A in figure (2) states<sup>5</sup>,  $\mathcal{G}^a(A) = \mathcal{E}_+^a(1) = E_+^a(1) + E_-^a(2) \approx 0$  in  $\mathcal{H}^p$ . Hence,  $\mathcal{U}_1$  completely decouples from  $\mathcal{H}^p$ . We now iterate the canonical transformation (6) with  $U_1, U_2$  replaced by  $U_{12}, U_3$ :

$$\mathcal{U}(2) \equiv U_{12}, \quad U_{123} \equiv U_{12} U_3, \quad (7)$$

$$\Rightarrow \mathcal{E}_+^a(2) = E_+^a(12) + E_-^a(3), \quad E_+^a(123) = E_+^a(3).$$

We get  $\mathcal{G}^a(B) = \mathcal{E}_+^a(2) = E_+^a(12) + E_-^a(3) = E_+^a(2) + E_-^a(3) \approx 0$  (Gauss law at B) and  $\mathcal{U}_2$  decouples. The last canonical transformation on the plaquette OABC is:

$$\begin{aligned} \mathcal{U}(3) &\equiv U_{123}, \quad \mathcal{W} \equiv U_{123} U(4), \\ \mathcal{E}_+^a(3) &= E_+^a(123) + E_-^a(4) \approx 0, \quad \mathcal{E}_+^a = E_+^a(4). \end{aligned} \quad (8)$$

Now  $\mathcal{U}_3$  decouples as  $\mathcal{G}^a(C) = \mathcal{E}_+^a(3)$ . Therefore, we need to focus only on the plaquette loop operators  $\{\mathcal{W}, \mathcal{E}_{\pm}^a\}$ . They are covariant under gauge transformation (15) at O. The SU(2) Gauss laws at A, B, C remove the strings  $\mathcal{U}_1, \mathcal{U}_2$  and  $\mathcal{U}_3$  respectively. The Gauss law at the origin, corresponding to the covariant transformations of the physical loop operators:

$$\mathcal{E}_{\mp} \rightarrow \Lambda_0 \mathcal{E}_{\mp} \Lambda_0^\dagger, \quad \mathcal{W} \rightarrow \Lambda_0 \mathcal{W} \Lambda_0^\dagger, \quad (9)$$

reduces to global constraints:

$$\mathcal{G}^a = \mathcal{L}^a \equiv \mathcal{E}_-^a + \mathcal{E}_+^a = 0. \quad (10)$$

It is convenient to describe SU(2) loop electric field and flux operators in terms of the prepotential doublets [8]:

$$\begin{aligned} \mathcal{E}_-^a &= (1/2) a^\dagger \sigma^a a, \quad \mathcal{E}_+^a = (1/2) b^\dagger \sigma^a b \\ \mathcal{W}_{\alpha\beta} &= \frac{1}{\sqrt{(N+1)}} \left( \tilde{a}_\alpha^\dagger b_\beta^\dagger - a_\alpha \tilde{b}_\beta \right) \frac{1}{\sqrt{(N+1)}}. \end{aligned} \quad (11)$$

Above  $\tilde{a}_\alpha \equiv \epsilon_{\alpha\gamma} a_\gamma$ . Like link electric fields [11], loop electric fields satisfy  $(\vec{\mathcal{E}}_-)^2 = (\vec{\mathcal{E}}_+)^2$  implying  $N_a = N_b \equiv N$ .

The transformations (6), (7) and (8) define the new strings, loop operators in terms of the old Kogut-Susskind link operators. These relations, being canonical, can also be inverted. They will enable us to rewrite Kogut-Susskind Hamiltonian in terms of loop operators in section 5. The loop flux operators are trivial to invert and we get:

$$\begin{aligned} U(1) &= \mathcal{U}_1, & U(2) &= \mathcal{U}_1^\dagger \mathcal{U}_2, \\ U(3) &= \mathcal{U}_2^\dagger \mathcal{U}_3, & U(4) &= \mathcal{U}_3^\dagger \mathcal{W}. \end{aligned} \quad (12)$$

<sup>4</sup>Note that the shapes of loops and strings will depend on the canonical transformations.

<sup>5</sup>  $A \approx 0$  implies that the operator A annihilates all physical states in  $\mathcal{H}^p$ .

The electric field relations are also easy to invert<sup>6</sup>:

$$\begin{aligned} E_-^a(1) &= \mathcal{E}_-^a(1) + \mathcal{E}_-^a(2) + \mathcal{E}_-^a(3) + \mathcal{E}_-^a, \\ E_-^a(2) &= R_{ab}(\mathcal{U}_1^\dagger) [\mathcal{E}_-^b(2) + \mathcal{E}_-^b(3) + \mathcal{E}_-^b], \\ E_-^a(3) &= R_{ab}(\mathcal{U}_2^\dagger) [\mathcal{E}_-^b(3) + \mathcal{E}_-^b], \\ E_-^a(4) &= R_{ab}(\mathcal{U}_3^\dagger) \mathcal{E}_-^b. \end{aligned} \quad (13)$$

Note that (a) the relations (12), (13) are consistent with the gauge transformation properties of the Kogut-Susskind link operators and the newly defined loop operators, (b) As expected, the inverse relations (13) transform the original Gauss law in terms of link operators at the origin,  $E_-^a(1) + E_+^a(4) = 0$ , into the Gauss law in terms of the loop operators<sup>7</sup> in (10),  $\mathcal{E}_-^a + \mathcal{E}_+^a = 0$ , in  $\mathcal{H}^p$ , (c) the gauge invariant operators, e.g., the quadratic electric field terms (see Kogut-Susskind Hamiltonian (26) in section 5), do not contain unphysical string variables. Further, we note that:

- Every lattice site (x,y) away from the origin (the corners A, B, C in this simple case) is attached with a string  $\mathcal{U}(x,y)$  ( $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$  in our case). They all start from the origin.
- The Gauss law operators at these sites are the right electric fields of the strings. In the single plaquette case:

$$\mathcal{G}^a(1,0) = \mathcal{E}_+^a(1), \mathcal{G}^a(1,1) = \mathcal{E}_+^a(2), \mathcal{G}^a(0,1) = \mathcal{E}_+^a(3),$$

making these strings devoid of any physical consequences.

The above features of the canonical transformations, converting links into physical loops and unphysical strings, are general and valid for any  $SU(N)$  in any space dimension.

### 3.2. $\mathcal{P}$ Plaquettes case

On a  $d = 2$  lattice with open boundary conditions, the canonical transformations transform  $\mathcal{L}$  Kogut-Susskind link operators into  $\mathcal{N}-1$  unphysical strings operator and  $\mathcal{P}$  physical loop operators. The degrees of freedom match as:  $\mathcal{L} = \mathcal{P} + (\mathcal{N} - 1)$ . The  $(\mathcal{N} - 1)$  strings  $\mathcal{U}(x,y)$ , attached to the lattice site  $(x,y)$  away from the origin, are along the oriented paths:  $(0,0) \rightarrow (x,0) \rightarrow (x,y)$ . They are unphysical and decouple as a consequence of Gauss laws.

<sup>6</sup> We have used canonical electric field relations (6), (7) and (8) to derive (13). As an example:

$$\begin{aligned} E_-^a(1) &\equiv -R_{ab}(U_1)E_+^b(1) = -R_{ab}(U_1) [\mathcal{E}_+^b(1) - E_-^b(2)] = \mathcal{E}_-^a(1) \\ -R_{ab}(U_2)E_+^b(2) &= \mathcal{E}_-^a(1) - R_{ab}(U_2) [\mathcal{E}_+^b(2) - E_-^b(3)] = \mathcal{E}_-^a(1) + \mathcal{E}_-^a(2) \\ -R_{ab}(U_3)E_+^b(3) &= \mathcal{E}_-^a(1) + \mathcal{E}_-^a(2) - R_{ab}(U_3) [\mathcal{E}_+^b(3) - E_-^b(4)] = \mathcal{E}_-^a(1) \\ &+ \mathcal{E}_-^a(2) + \mathcal{E}_-^a(3) - R_{ab}(\mathcal{W})E_+^b(4) = \mathcal{E}_-^a(1) + \mathcal{E}_-^a(2) + \mathcal{E}_-^a(3) + \mathcal{E}_-^a. \end{aligned}$$

Similarly the other three relations in (13) can be derived.

<sup>7</sup>We have used  $E_-^a(1) \cong \mathcal{E}_-^a$  as  $\mathcal{E}(I) = 0$ ,  $I = 1, 2, 3$  in  $\mathcal{H}^p$  and  $E_+^a(4) \equiv -R_{ab}(U^\dagger(4))E_-^b(4) = -R_{ab}(\mathcal{W}^\dagger)\mathcal{E}_-^b = \mathcal{E}_+^a$ .

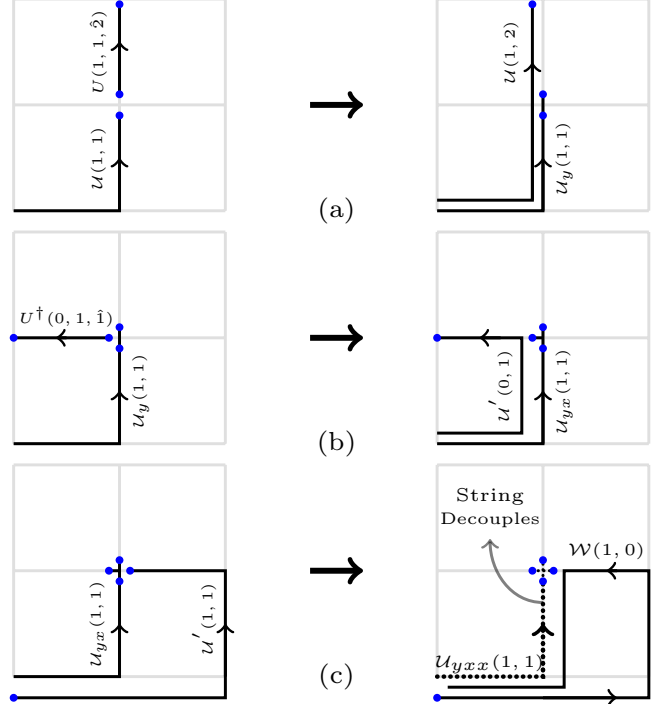


Figure 3: Formation of plaquette loops and the decoupling of strings. The loop plaquette operator  $\mathcal{W}(1,0)$  is shown. The  $\bullet$  represent the electric fields of the links, strings and loops. In the last figure, the right electric field of the string operator  $\mathcal{U}_{yxx}(1,1)$  is the Gauss law at  $(1,1)$ :  $\mathcal{U}_{yxx+}^a(1,1) = \mathcal{G}^a(1,1)$ .

As an example, we show in figure (3) how the final string attached to the lattice site  $(1,1)$  decouples. In figure (3-a), we attach the Kogut-Susskind link operator  $U(1,1;\hat{2})$  to the string  $\mathcal{U}(1,1)$  to obtain the new strings  $\mathcal{U}_y(1,1)$  and  $\mathcal{U}(1,2)$ . The subscript  $y$  of  $\mathcal{U}_y(1,1)$  shows that it has been extended in the  $y$  direction and its right electric field at  $(1,1)$  is the sum of two Kogut-Susskind electric fields in  $y$  directions. These two Kogut-Susskind electric fields are shown by the two blue dots at  $(1,1)$  in the right hand side of figure (3-a). We now extend the string  $\mathcal{U}_y(1,1)$  in the -ve  $x$  direction by attaching it with the Kogut-Susskind link operator  $U^\dagger(0,1,\hat{1})$  as shown in figure (3-b). This results in new strings  $\mathcal{U}_{yx}(1,1)$  and  $\mathcal{U}'(0,1)$  as shown in the right hand side of figure (3-b). Now the right electric field of  $\mathcal{U}_{yx}(1,1)$  at  $(1,1)$  is the sum of three Kogut-Susskind electric fields at this point as shown by the three blue dots in the right hand side of figure (3-b). We finally extend  $\mathcal{U}_{yx}(1,1)$  in the +ve  $x$  direction by attaching it with  $U'(1,1)$  as shown in the left hand side of figure (3-c). The result is  $\mathcal{U}_{yxx}(1,1)$  and the plaquette loop operator  $\mathcal{W}(1,0)$ . The right electric field of  $\mathcal{U}_{yxx}(1,1)$  is now sum of all four Kogut-Susskind electric fields at  $(1,1)$  as shown by the four blue dots in the right hand side of figure (3-c). The Gauss law implies that the string  $\mathcal{U}_{yxx}(1,1)$  excitations are outside  $\mathcal{H}^p$ . Therefore it can be ignored.

From now onwards, we focus on the remaining physical  $\mathcal{P}$  loop operators denoted by  $\mathcal{W}(x,y)$ . They are along the paths:  $(0,0) \rightarrow (x,0) \rightarrow (x,y) \rightarrow (x+1,y) \rightarrow (x+1,y+1) \rightarrow (x,y+1) \rightarrow (x,0)$ .

$1) \rightarrow (x, y+1) \rightarrow (x, 0) \rightarrow (0, 0)$ . These paths are shown in figure (5) for a small lattice with  $N = 2$ . It is convenient to collectively denote them by  $\mathcal{W}(p)$ ,  $p = 1, 2, \dots, \mathcal{P}$ . The physical SU(2) prepotentials are defined as:

$$\begin{aligned} \mathcal{E}_-^a(p) &= \frac{1}{2} a^\dagger(p) \sigma^a a(p), \quad \mathcal{E}_+^a(p) = \frac{1}{2} b^\dagger(p) \sigma^a b(p) \\ \mathcal{W}_{\alpha\beta}(p) &= \frac{1}{\sqrt{(N(p)+1)}} \left( \mathcal{W}_{\alpha\beta}^+(p) - \mathcal{W}_{\alpha\beta}^-(p) \right) \frac{1}{\sqrt{(N(p)+1)}}. \end{aligned} \quad (14)$$

In (14),  $\mathcal{W}_{\alpha\beta}^+(p) \equiv \tilde{a}_\alpha^\dagger(p) b_\beta^\dagger(p)$  and  $\mathcal{W}_{\alpha\beta}^-(p) \equiv a_\alpha(p) \tilde{b}_\beta(p)$ . All loop operators in (14) are invariant under gauge transformations anywhere on the lattice except origin. Under gauge transformations at the origin, they all transform together like adjoint matter fields:

$$\begin{aligned} \mathcal{E}_\mp(p) &\rightarrow \Lambda_0 \mathcal{E}_\mp(p) \Lambda_0^\dagger, \quad p = 1, 2, \dots, \mathcal{P}, \\ \mathcal{W}(p) &\rightarrow \Lambda_0 \mathcal{W}(p) \Lambda_0^\dagger; \quad p = 1, 2, \dots, \mathcal{P}. \end{aligned} \quad (15)$$

All  $2\mathcal{P}$  prepotential creation operators transform as matter doublets:

$$a_\alpha(p) \rightarrow (\Lambda_0)_{\alpha\beta} a_\beta(p), \quad b_\alpha(p) \rightarrow (\Lambda_0)_{\alpha\beta} b_\beta(p). \quad (16)$$

The Gauss law at the origin reduces to simple constraints:

$$\mathcal{G}^a = \sum_{p=1}^{\mathcal{P}} \mathcal{L}^a(p) = \sum_{p=1}^{\mathcal{P}} \left( \mathcal{E}_-^a(p) + \mathcal{E}_+^a(p) \right) \equiv \mathcal{L}_{total}^a \approx 0. \quad (17)$$

This is a straightforward generalization of the single plaquette result (9). Thus after these canonical transformations, the nontrivial issue of non-abelian gauge invariance reduces to a much simpler issue of global invariance. We solve these constraints in the next section to construct a basis in  $\mathcal{H}^P$ .

#### 4. Loop States & Hydrogen Atoms

We first start with the simple single plaquette case. We identify the hydrogen atom angular momentum, Lenz vector operators with the SU(2) loop electric field operators of the gauge theory:

$$J_\mp^a \leftrightarrow \mathcal{E}_\mp^a.$$

This identification further implies:  $L^a \leftrightarrow \mathcal{L}^a$  where  $L^a$  are the angular momentum operators of a hydrogen atom and  $\mathcal{L}^a$  are the generators of Gauss law (10) in a single plaquette case. This immediately implies that CSCO-I and CSCO-II of hydrogen atom also characterize lattice gauge theory Hilbert space. The CSCO-II is even more natural for gauge theory as the remaining three SU(2) Gauss law constraints at the origin (10) are trivially removed in this coupled basis. In the single plaquette case, the non-abelian

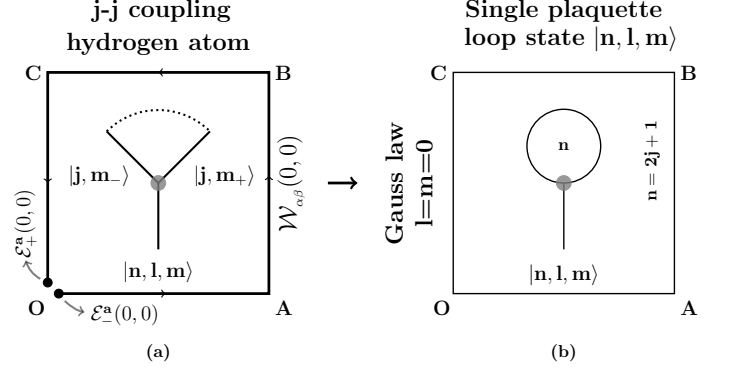


Figure 4: A graphical tadpole representation of hydrogen atom states  $|n \ l \ m\rangle$  or equivalently a SU(2) loop state over a plaquette. The dotted arch represents  $j_+ = j_- = j$  in the j-j coupling (3) denoted by  $\bullet$ . The tadpole loop represents the SU(2) flux circulating within a plaquette. The vertical leg of tadpole represents leakage of angular momentum flux  $(l, m)$  through the plaquette.

Gauss laws (10) state that the spherically symmetric hydrogen atom states  $|n \ l = 0 \ m = 0\rangle$  form an orthonormal loop basis in  $\mathcal{H}^P$ :

$$\begin{aligned} |n\rangle &\equiv |n, l = 0, m = 0\rangle = \frac{(k_+)^n}{\sqrt{n!(n+1)!}} |0\rangle. \\ \sum_{n=1}^{\infty} |n\rangle \langle n| &= \mathcal{I}, \quad \langle m|n\rangle = \delta_{nm}. \end{aligned} \quad (18)$$

In (18)  $\mathcal{I}$  is the identity operator in  $\mathcal{H}^P$ . The loop creation-annihilation and number operators are defined as:  $k_+ \equiv a^\dagger \cdot \tilde{b}^\dagger$ ,  $k_- \equiv a \cdot \tilde{b}$  and  $k_0 \equiv 1/2(N_a + N_b + 2)$ . They satisfy SU(1,1) algebra:  $[k_-, k_+] = 2k_0$ ,  $[k_0, k_\pm] = \pm k_\pm$ .

At this stage, before going to  $\mathcal{P}$  plaquette case, it is convenient to graphically represent the coupled state  $|n \ l \ m\rangle$  in (3) on a plaquette by a tadpole diagram as shown in figure (4-a,b). The loop at the top of the tadpole in a plaquette represents the non-abelian flux circulating in a loop within the plaquette. The vertical leg of the tadpole, on the other hand, represents the leakage of non-abelian flux through the plaquette. More precisely, the anti-symmetric part of the coupling between  $a^\dagger$  and  $b^\dagger$  (Young tableau boxes arranged vertically in pairs) present in  $|j, m_- \rangle$  and  $|j, m_+ \rangle$  states is shown by the circle. The leg represents the symmetric part of this coupling (Young tableau boxes arranged along a row). We now draw tadpoles over each of the  $\mathcal{P}$  plaquettes and then couple their emerging angular momentum fluxes  $(l_p, m_p)$  with  $p = 1, \dots, \mathcal{P}$  in a sequential manner as in figure (5-b). It corresponds to going from decoupled tadpole basis diagonalizing  $3\mathcal{P}$  CSCO-II operators  $\{J_p^2, L_p^2, L_p^{a=3}\}$ , with eigenvalues  $(j_p(j_p+1), l_p(l_p+1), m_p)$ , to a coupled basis which diagonalizes the following coupled angular momentum opera-

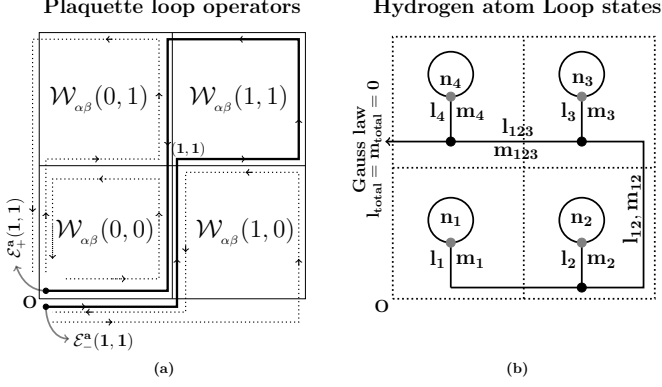


Figure 5: [a] The construction of the plaquette loop operators  $\mathcal{W}_{\alpha\beta}(x, y)$  and  $\mathcal{E}_{\mp}^a(x, y)$  in a four plaquette case. [b] The SU(2) hydrogen atom/tadpole basis with hydrogen atom states  $|n l m\rangle$ . The  $\bullet$  and  $\circ$  in [b] represent  $jj$  and  $ll$  couplings in (3) and (20) respectively.

tors:

$$\left\{ \begin{array}{cccc} \vec{J}_1^2 & \vec{J}_2^2 & \cdots \vec{J}_{\mathcal{P}-1}^2 & \vec{J}_{\mathcal{P}}^2 \\ \vec{L}_1^2 & \vec{L}_2^2 & \cdots \vec{L}_{\mathcal{P}-1}^2 & \vec{L}_{\mathcal{P}}^2 \\ (\vec{L}_{12})^2 & (\vec{L}_{123})^3 & \cdots (\vec{L}_{total})^2 & \vec{L}_{total}^{a=3} \end{array} \right\}.$$

Above  $(\vec{L}_{1,2,\dots,q})^2 \equiv (\vec{L}_1 + \vec{L}_2 + \cdots \vec{L}_q)^2$  with eigenvalue  $l_{1,2,\dots,q}(l_{1,2,\dots,q} + 1)$  and  $q = 2, 3, \dots, \mathcal{P}$ . We have put  $\vec{L}_{total} = 0$  because of the Gauss law (17). This further implies that  $l_{1,2,\dots,\mathcal{P}-1} = l_{\mathcal{P}}$  as  $\vec{L}_{total} \equiv (\vec{L}_{1,2,\dots,\mathcal{P}-1} + \vec{L}_{\mathcal{P}}) = 0$ . The resulting orthonormal and complete loop basis is:

$$\left\{ \begin{array}{cccc} n_1 & n_2 & \cdots & n_{\mathcal{P}} \\ l_1 & l_2 & \cdots & l_{\mathcal{P}} \\ l_{12} & l_{123} & \cdots & l_{123\cdots\mathcal{P}-2} \end{array} \right\} \equiv \left\{ |n_1 l_1 m_1\rangle \otimes |n_2 l_2 m_2\rangle \cdots \otimes |n_{\mathcal{P}} l_{\mathcal{P}} m_{\mathcal{P}}\rangle \right\}_{m_{total}=0}^{l_{total}=0}$$

Wigner coupled states of hydrogen atoms

(19)

The above loop basis (shown in figure (5-b) for 4 plaquette case) will be briefly denoted<sup>8</sup> by  $|[n] [l] [ll]\rangle$ . The symbols  $[n]$ ,  $[l]$  and  $[ll]$  stand for the sets  $(n_1 \cdots n_{\mathcal{P}})$ :  $\mathcal{P}$  principle quantum numbers;  $[l_1 \cdots l_{\mathcal{P}}]$ :  $\mathcal{P}$  angular momentum quantum numbers and  $(l_{12}, l_{123}, \dots, l_{123\cdots(\mathcal{P}-2)})$ :

<sup>8</sup> As an example the loop states over 4 plaquettes in figure (5-b) are constructed as;

$$|[n] [l] [ll]\rangle = \sum_{m_1 m_2 m_3 m_4 m_{12}} C_{l_1 m_1, l_2 m_2}^{l_{12} m_{12}} C_{l_{12} m_{12}, l_3 m_3}^{l_{123} m_{123}} C_{l_{123} m_{123}, l_4 m_4}^{0 0} |n_1 l_1 m_1\rangle |n_2 l_2 m_2\rangle |n_3 l_3 m_3\rangle |n_4 l_4 m_4\rangle. \quad (20)$$

The states  $|n l m\rangle$  are constructed in (2), (3) and figure (4). They are orthonormal as well as complete:

$$\langle [\bar{n}] [\bar{l}] [\bar{ll}] | [n] [l] [ll] \rangle = \delta_{[\bar{n}], [n]} \delta_{[\bar{l}], [l]} \delta_{[\bar{ll}], [ll]} \quad (21)$$

$$\sum_{[n]} \sum_{[l]} \sum_{[ll]} | [n] [l] [ll] \rangle \langle [n] [l] [ll] | = \mathcal{I}. \quad (22)$$

The operator  $\mathcal{I}$  denotes the identity operator in  $\mathcal{H}^{\mathcal{P}}$ .

$(\mathcal{P} - 3)$  coupled angular momentum quantum numbers respectively. Thus the hydrogen atom loop basis (19) in  $\mathcal{H}^{\mathcal{P}}$  is labelled by  $\mathbb{N} = 3(\mathcal{P} - 1)$  gauge invariant quantum numbers. As expected, this is also the dimension of quotient space  $\mathbb{N} = \left[ \frac{\otimes_{links} SU(2)}{\otimes_{sites} SU(2)} \right]$  in  $d = 2$ .

We now analyze this equivalence in the dual magnetic description. We again start with single plaquette basis  $|j m_- m_+\rangle$  in (2) and make a duality transformation to define states on SU(2) group manifold  $S^3$  as:

$$|\Omega_W\rangle = \sum_{j=0}^{\infty} \sum_{m_{\mp}=-j}^{+j} \{j\} D_{m_- m_+}^j(\Omega_W) |j, m_-, m_+\rangle \quad (23)$$

In (23)  $\{j\} \equiv (2j + 1)^{\frac{1}{2}}$ ,  $D_{m_- m_+}^j(\Omega_W)$  are the Wigner matrices characterized by SU(2) group manifold  $S^3$ :

$$\Omega_W(w_0, \vec{w}) \equiv w_0 \sigma_0 + i \vec{w} \cdot \vec{\sigma}, \quad w_0^2 + \vec{w}^2 = 1 : S^3.$$

The recursion relations of Wigner matrices show [12] that the orthonormal and complete angular states (23) also diagonalize the plaquette loop operators  $\mathcal{W}_{\alpha\beta}$  in (11):

$$\mathcal{W}_{\alpha\beta} |\Omega_W\rangle = (\Omega_W(\omega_0, \vec{w}))_{\alpha\beta} |\Omega_W\rangle. \quad (24)$$

Under global SU(2) transformation (15):

$$|\Omega_W\rangle \rightarrow |\Lambda_o \Omega_W \Lambda_o^\dagger\rangle. \quad (25)$$

The gauge generators  $L^1, L^2, L^3$  in (10) rotate  $(w_2 w_3), (w_3 w_1), (w_1 w_2)$  planes respectively leaving  $w_0$  (gauge) invariant. Defining “Lenz operators” in lattice gauge theory as  $A^a \equiv \mathcal{E}_+^a - \mathcal{E}_-^a$ , we see that  $(A^1, A^2, A^3)$  generate rotations in  $(w_0 w_1), (w_0 w_2), (w_0 w_3)$  planes respectively. Therefore, the actions of  $(L^a, A^a)$  on  $\Omega_W$  in gauge theory is exactly same as the actions of  $(L^a, A^a)$  on  $\Omega_H$  in hydrogen atom. Therefore, we further identify:

$$\Omega_H \sim \Omega_W \equiv \Omega.$$

## 5. SU(2) Loop Dynamics & Loop Operators

We now construct non-abelian loop dynamics directly in terms of the loop operators  $\mathcal{W}_{\alpha\beta}(x, y)$  and their conjugate loop electric fields  $\mathcal{E}_{\mp}^a(x, y)$  constructed in section (3). We start with the Kogut Susskind Hamiltonian [11]:

$$H = g^2 \sum_l \vec{E}_l^2 + \frac{K}{g^2} \sum_p (2 - \text{Tr } U_p). \quad (26)$$

In (26)  $K$  is a constant,  $l \equiv (x, y, \hat{i})$  denotes a link in  $\hat{i}$  direction,  $p$  denotes a plaquette and  $\text{Tr } U_p \equiv \text{Tr } (U_1 U_2 U_3^\dagger U_4^\dagger)$  is the magnetic field term on a plaquette  $p$ . In the simple single plaquette case, the Kogut-Susskind Hamiltonian (26) in terms of the loop operators is:

$$H = 4g^2 \mathcal{E}^2 - \frac{K}{g^2} (2 - \text{Tr } \mathcal{W}). \quad (27)$$

In going from link description (26) to loop description (27) we have used (8) and (13). The first term  $4\vec{E}^2 (\equiv 4\mathcal{E}_-^2 = 4\mathcal{E}_+^2)$  describes the loop electric fields. The factor 4 is because of the 4 links on the plaquette and the relations (13) after putting the three string electric fields  $\mathcal{E}_\mp^a(1) = \mathcal{E}_\mp^a(2) = \mathcal{E}_\mp^a(3) \cong 0$ . The Hamiltonian (27) is invariant under global SU(2) transformations (9).

Similarly, the Kogut Susskind Hamiltonian in the loop space on a finite lattice is [12]:

$$H = \sum_{(x,y)} \left\{ g^2 \vec{E}^2(x, y, \hat{1}) + g^2 \vec{E}^2(x, y, \hat{2}) + \frac{K}{g^2} \left( 2 - \text{Tr } \mathcal{W}(x, y) \right) \right\}. \quad (28)$$

In (28),  $E^2(x, y, \hat{i}) = E_-^2(x, y, \hat{i}) = E_+^2(x, y, \hat{i})$  as discussed in section 3 and we have used the property of the canonical transformations:  $\text{Tr } U_p(x, y) = \text{Tr } (\mathcal{U}^\dagger(x, y) \mathcal{W}(x, y) \mathcal{U}(x, y)) = \text{Tr } \mathcal{W}(x, y)$ , for converting the magnetic field term. The Kogut-Susskind electric fields above are given by [12]:

$$\begin{aligned} E_-^a(x, y, \hat{1}) &\cong R_{ab}(\mathcal{U}(x, y)) \left[ \mathcal{E}_-^a(x, y) + \mathcal{E}_+^a(x, y-1) + \Delta_X^a \right] \\ E_-^a(x, y, \hat{2}) &\cong R_{ab}(\mathcal{U}(x, y)) \left[ \mathcal{E}_+^b(x, y) + \mathcal{R}_w(x, y)_{bc} \mathcal{E}_-^c(x-1, y) + \Delta_Y^b \right] \end{aligned} \quad (29)$$

In (29),  $E_-^a(x, y, \hat{i})$  is the electric field at  $(x, y)$  in the  $\hat{i}$  direction and rotates the link operator  $U(x, y, \hat{i})$  from the left as in (5).  $\Delta_X, \Delta_Y$  and  $\mathcal{R}_w$  are defined as:

$$\begin{aligned} \Delta_X^a(x, y) &\equiv \delta_{y,0} \sum_{r=x+1}^{\mathcal{N}} \sum_{s=0}^{\mathcal{N}} \mathcal{L}^a(r, s), \quad \Delta_Y^a(x, y) \\ &\equiv \sum_{s=(y+1)}^{\mathcal{N}} \mathcal{L}^a(x, s), \quad \mathcal{R}_w(x, y) \equiv \prod_{q=0}^{y-1} R(\mathcal{W}(x-1, q)) \end{aligned} \quad (30)$$

Note that the physical loop operators  $\mathcal{E}_\mp^a(x, y)$ ,  $\mathcal{W}(x, y)$  are located at  $(x, y)$  as shown in figure (5-a). There are no gauge fields. The relations (29) are generalizations of the simple single plaquette results in (13). Like in single plaquette case these relations are arrived at by systematically inverting the canonical transformations at each stage through out the lattice [12]. Again, as expected, the unphysical strings  $\mathcal{U}(x, y)$  attached to sites  $(x, y)$  in (29) disappear from the Hamiltonian (28) as  $R^T R = 1$ .

The Hamiltonian (28) describes dynamics directly in terms of the essential loop creation, annihilation operators and their conjugate electric fields. There is no local gauge invariance and no gauge fields. The gauge transformations at the origin manifest themselves in the global SU(2) invariance (15) of the Hamiltonian (28).

We note that in going from links to loops ((26) to (28)), all interactions have shifted from the  $(1/g^2)$  magnetic field

term to  $(g^2)$  electric field term. Therefore, as opposed to strong coupling ( $g^2 \rightarrow \infty$ ) expansion with simple  $g^2 E^2$  term in (26), the loop formulation (28) with the simple  $1/g^2 \text{Tr } \mathcal{W}$  term provides an alternative framework appropriate to develop gauge invariant weak coupling ( $g^2 \rightarrow 0$ ) expansion near the continuum limit<sup>9</sup>. In fact, the magnetic field term  $1/g^2 \text{Tr } \mathcal{W} \sim 1/g^2 (k_+ + k_-)$  dominating in this limit, has the simplest possible action on the hydrogen atom loop basis (19). Using (14) and (19), the matrix elements are:

$$\begin{aligned} &\langle [\bar{n}] [\bar{l}] [\bar{l}] | \text{Tr } \mathcal{W}(p) | [n] [l] [ll] \rangle \\ &= \delta_{[\bar{l}], [l]} \delta_{[\bar{l}], [l]} \delta_{[\bar{n}], [n]'} \left[ \delta_{\bar{n}_p, n_p+1} + \delta_{\bar{n}_p, n_p-1} \right]. \end{aligned} \quad (31)$$

Above  $n_p = 2j_p + 1$  and  $\bar{n}_p \equiv 2\bar{j}_p + 1$  and  $\delta_{[\bar{n}], [n]'}$  denotes the delta function over  $(\mathcal{P} - 1)$  principal quantum numbers over all plaquettes  $p' \neq p$ . This should be compared with the action of the standard magnetic field term  $1/g^2 \text{Tr } (U_1 U_2 U_3^\dagger U_4^\dagger)$  on the standard SU(2) spin network basis describing orthonormal loop states [8]. This action is extremely complicated and the matrix elements are given by 18-j (30-j) Wigner coefficients in  $d = 2$  ( $d = 3$ ) dimension as compared to the simple  $\delta$  function over a single principal quantum number in (31).

All interactions in the Hamiltonian (28) in this loop formulation are contained in (29). They describe nearest neighbour loop-loop interactions through their electric fields along with non-local terms  $\Delta_X(x, y)$ ,  $\Delta_Y(x, y)$ ,  $\mathcal{R}_w(x, y)$ . However, these non-local interactions can be ignored at tree level near the continuum ( $g^2 \rightarrow 0$ ) limit as in this limit  $\mathcal{L}^a(x, y) \equiv \mathcal{E}_+^a(x, y) + \mathcal{E}_-^a(x, y) \rightarrow 0$  and  $R_{ab}(\mathcal{W})(x, y) \rightarrow \delta_{ab}$ . Thus we have a generalized “spin model” with nearest neighbour interactions along with spin creation ( $k_+$ ) and spin annihilation ( $k_-$ ) terms. The analysis of this model through variational method and coupled cluster expansion is under study and will be reported elsewhere.

## 6. Summary & Discussion

In this work we have obtained a loop formulation of pure SU(2) lattice gauge theory through a series of canonical transformations. These canonical transformations, constructed in section (3), are also valid for any SU(N) group. They can also be extended to higher dimension. As shown in this work the problem of over-completeness of SU(N) loop states leading to SU(N) Mandelstam constraints amongst Wilson loops is bypassed in this approach. To the best of our knowledge the solutions of SU(N) Mandelstam constraints for  $N \geq 3$  do not exist

<sup>9</sup>In strong coupling ( $g^2 \rightarrow \infty$ ) expansion non-interacting terms  $g^2 E_{link}^2$  are trivially diagonalized and 4 flux interaction terms  $1/g^2 \text{Tr}(U_1 U_2 U_3^\dagger U_4^\dagger)$  are treated in perturbation. However, one is far away from continuum.

[10]. The  $SU(N)$  canonical transformations discussed in this work also provide a complete set of mutually independent  $SU(N)$  Wilson loop operators.

In the case of  $SU(2)$ , a gauge invariant state can be expanded in the hydrogen atom basis. Therefore, any gauge invariant operator acting on a physical state generates transitions in the underlying basis states  $|n\ l\ m\rangle$ . Such transitions are produced by the generators of the dynamical symmetry group  $SO(4,2)$  of hydrogen atom. Therefore,  $SU(2)$  loop Hamiltonian can also be described in terms of  $SO(4,2)$  generators. These results will be reported later.

Having removed all redundant gauge and loop degrees of freedom through canonical transformations, we should further identify the relevant and interesting part of  $\mathcal{H}^P$  for low energy physics. This can be done using tensor networks or matrix product ideas [7] as the  $3(\mathcal{P} - 1)$  gauge invariant quantum numbers can be thought of as generalized spins over plaquettes of the lattice. In context of cold atoms, imposing non-trivial and exotic non-abelian Gauss law constraints experimentally at every lattice site is a challenging task. Different ways of dealing with these local constraints have been proposed in the past [14]. The present formulation with hydrogen atoms naturally solve these constraints and may be useful in the cold atom experiments in the future. This absence of local Gauss laws should also help us define entanglement entropy of two complimentary regions in a gauge invariant state bypassing obstacles [15] created by them at the boundary of the two regions. The two regions can have mutually independent hydrogen atom/tadpole basis which are joined together across the boundary at the end.

*Acknowledgments:* MM thanks Ramesh Anishetty and H. S. Sharatchandra for useful discussions. MM especially acknowledges H S Sharatchandra for introducing him to the canonical transformations which were crucial for this work and for initial discussions on the resulting Hamiltonian. TPS thanks CSIR for financial support.

## References

- [1] K. G. Wilson, Phys. Rev. D 10 (1974) 2445 ; S. Mandelstam, Phys. Rev. D 19 (1979) 2391 ; T. T. Wu, C. N. Yang, Phys. Rev. D 12 (1975) 3845 .
- [2] W. Pauli, Z. Physik 36 (1926) 336-363; V. Fock , Z. Phys. 98 (1935) 145 , For English translation see S. F. Singer, *Linearity, Symmetry, and Prediction in the Hydrogen Atom* (Springer 2000) , p. 286 ; V. Bargmann , Z. Phys. 99 (1936) 576 ; M. Bander, C. Itzykson, Rev.Mod.Phys. 38 (1966) 346;
- [3] Brian G. Wybourne, *Classical Groups for Physicists* (John Wiley & Sons 1974), see chapter 21.
- [4] C. Rovelli, *Quantum Gravity* (Cambridge Univ. Press 2004); A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244 .
- [5] E. Fradkin, *Field Theories of Condensed Matter Physics* (Cambridge University Press, 2013).
- [6] E. Zohar, J. I. Cirac, B. Reznik , Phys. Rev. Lett. 110 (2013) 125304, E. Zohar, J. I. Cirac, B. Reznik, Phys. Rev. Lett. 109, 125302(2012); L. Tagliacozzo, A. Celi, P. Orland, M. Lewenstein, Nature Commun. 4 (2013) 2615 ; D. Banerjee et. al. Phys. Rev. Lett. 110 (2013) 125303. E. Zohar , M. Burrello, Phys. Rev. D 91 (2015) 054506 ; E. Zohar, J. I. Cirac, B. Reznik, arXiv:1503.02312 (2015);
- [7] S. Östlund, S. Rommer, Phys. Rev. Lett. 19 (1995) 3537 ; S. Singh, G. Vidal, Phys. Rev. B 86 (2012) 195114 ; T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, arXiv:1505.04440 (2015); S. Kühn, E. Zohar, J. I. Cirac, and M. C. Bañuls, arXiv:1505.04441 (2015 )
- [8] M. Mathur, Nucl. Phys. B 779 (2007) 32 ; M. Mathur, J. Phys. A 38 (2005) 10015-10026; R. Anishetty, I. Raychowdhury, Phys. Rev. D 90 (2014) 114503.
- [9] R. Anishetty, H. S. Sharatchandra, Phys. Rev. Lett. 65 (1990) 81; H. S. Sharatchandra, Nucl. Phys. B 196 (1982) 62 , D. Robson, D. M. Weber, Z. Phys. C 15 (1982) 199 .
- [10] A.A. Migdal , Phys. Repts. 102 (1983) 199 ; V.F Muller , W. Ruhl, Nucl. Phys. B. 230 (1984) 49; W. Furmanski and A. Kolawa, Nuclear Physics B. 291 (1987) 594; Bernd Brugmann, Phys. Rev. D 43 (1991) 567 ; R. Loll, Nucl. Phys. B 400 (1993) 126 ; N.J. Watson, Phys. Letts. B 323 (1994) 385 ; R. Gambini, J. Pullin *Loops, knots, gauge theories and Quantum gravity*, Cambridge University Press (1996); N.E. Ligterink, N.R. Walet, R.F. Bishop, Ann of phys. 284 (2000) 215;
- [11] J. Kogut, L. Susskind, Phys. Rev. D 11 (1975) 395.
- [12] Manu Mathur, T. P. Sreeraj, Under preparation.
- [13] D. A. Varshalovich et al., *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
- [14] K. Stannigel, P. Hauke, D. Marcos, M. Hafezi, S. Diehl, M. Dalmonte, P. Zoller, Phys. Rev. Lett. 112 (2014) 120406; E. Zohar, E. Cirac, B Reznik, Phys. Rev. A 88 (2013) 023617.
- [15] Horacio Casini, Marina Huerta, and Jos Alejandro Rosabal, Phys. Rev. D 89 (1991) 085012 ;